Universidade do Estado do Rio de Janeiro
2^a prova de Física Matemática II
28 de novembro 2017, 8:50h

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1. **(3.5)**
Calcule os primeiros termos (até primeira ordem) da série de potências da função:

$$f(z) = \frac{e^z}{z(z+1)^2}$$

a) em potências de $(z+1)$, (indique claramente a região de convergência).

b) em potências de z, para $|z| < 1$.

2. **(3.5)**
Usando o teorema dos resíduos calcule a integral:

$$\oint_C \frac{dz}{z^3(z+4)}$$

onde

a) $C : |z| = 2$

b) $C : |z+2| = 3$

(os caminhos estão orientados no sentido anti-horário.)

3. **(3.0)**
Usando técnicas de variável complexa, calcule a integral real:

$$\int_0^\infty \frac{x^2}{(x^2+4)^2} \; dx$$

Justifique **rigorosamente** (explicando em detalhe) cada passo realizado para sua solução.

P2 FIS. MAT II

Ex 1

$$f(z) = \frac{e^{z}}{z(z+1)^2}$$

a) $$f(z) = \frac{1}{(z+1)^2}\left\{ e^{z} \times \frac{1}{z} \right\} = \frac{1}{(z+1)^2}\left\{ e^{z+1-1} \times \frac{1}{z+1-1} \right\}$$

$$f(z) = -\frac{1}{e(z+1)^2}\left\{ e^{z+1} \times \frac{1}{1-(z+1)} \right\}$$

$$= -\frac{1}{e(z+1)^2}\left(1 + (z+1) + \frac{1}{2!}(z+1)^2 + \frac{1}{3!}(z+1)^3 + \cdots \right) \times$$

$$\times \left(1 + (z+1) + (z+1)^2 + (z+1)^3 + \cdots \right)$$

$$f(z) = -\frac{1}{e(z+1)^2}\left\{ 1 + (z+1) + (z+1)^2 + (z+1)^3 + (z+1) + (z+1)^2 \right.$$

$$\left. + (z+1)^3 + \frac{1}{2!}(z+1)^2 + \frac{1}{2!}(z+1)^3 + \frac{1}{3!}(z+1)^3 + O((z+1)^4) \right\}$$

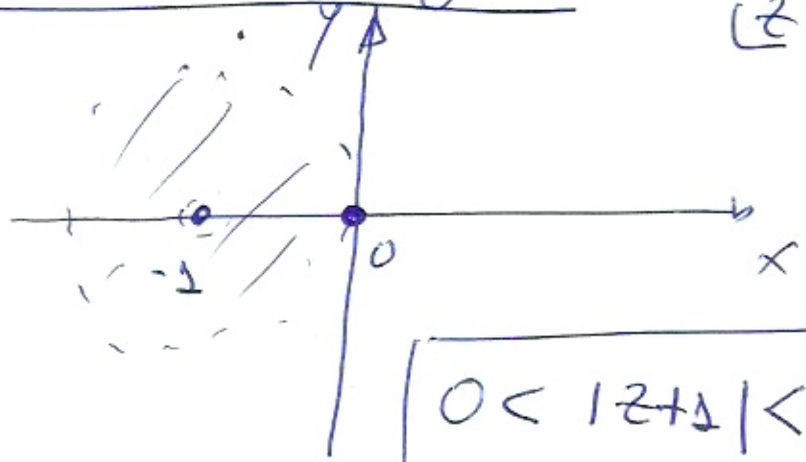
$$\cancel{f(z) = -\frac{1}{e(z+1)^2} - \frac{1}{e}}$$

$$f(z) = -\frac{1}{e(z+1)^2}\left\{ 1 + 2(z+1) + \left(2+\frac{1}{2}\right)(z+1)^2 + \left(1+\frac{1}{2}+\frac{1}{6}\right)(z+1)^3 + \cdots \right\}$$

$$f(z) = -\frac{1}{e(z+1)^2}\left\{ 1 + 2(z+1) + \frac{5}{2}(z+1)^2 + \frac{5}{3}(z+1)^3 + \cdots \right\}$$

$$\boxed{f(z) \sim -\frac{1}{e(z+1)^2} - \frac{2}{e}\frac{1}{(z+1)} - \frac{5}{2e} - \frac{5}{3e}(z+1) + \cdots}$$

Região de Convergência



$$\boxed{0 < |z+1| < 1}$$

b)

$$f(z) = \frac{1}{z}\left\{ e^{z} \times \frac{1}{(z+1)^2} \right\}$$

série de Taylor

$$e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\frac{1}{(1+z)^2} = 1 - 2z + 3z^2 - 4z^3 - \cdots$$

$$f(z) = \frac{1}{z}\left(1 + z + \frac{z^2}{2!} + \cdots\right)\left(1 - 2z + 3z^2 - \cdots\right)$$

$$= \frac{1}{z}\left\{1 - 2z + 3z^2 + z - 2z^2 + \frac{z^2}{2!} + \cdots\right\}$$

$$f(z) = \frac{1}{z}\left\{1 - z + \frac{3}{2}z^2 + \cdots\right\}$$

$$\boxed{f(z) = \frac{1}{z} - 1 + \frac{3}{2}z + \cdots}$$

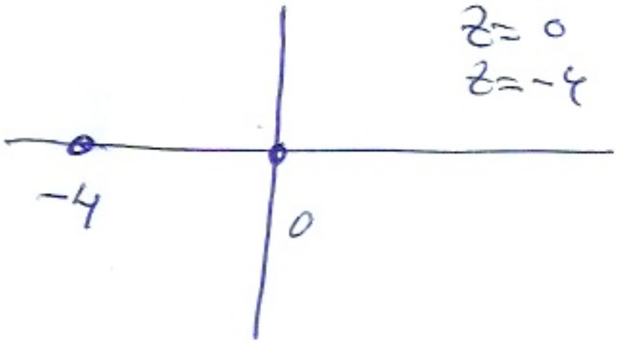
Ex 2

$$\oint_C \frac{dz}{z^3(z+4)}$$

a) $f(z) = \dfrac{1}{z^3(z+4)}$

singularidades

$z = 0$
$z = -4$



$z=-4$ polo simples

$z = 0$ polo triplo

$$\operatorname{Res} f(-4) = \lim_{z \to -4} \frac{1}{z^3} = -\frac{1}{4^3}$$

$$\operatorname{Res} f(0) = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2}\left(\frac{1}{z+4}\right) = \frac{1}{2!} \times \frac{2}{4^3} = \frac{1}{4^3}$$

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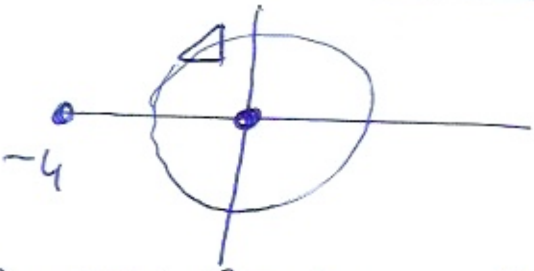
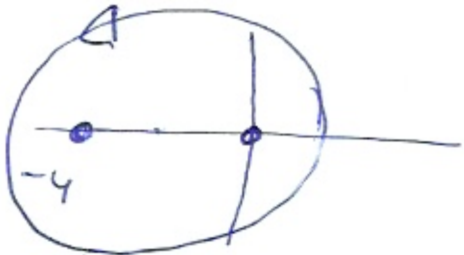
a) $|z|=2$



$$\oint_C f(z)\,dz = 2\pi i \operatorname{Res} f(0) = \frac{2\pi i}{4^3} = \boxed{\frac{\pi i}{32}}$$

b) $|z+2|=3$



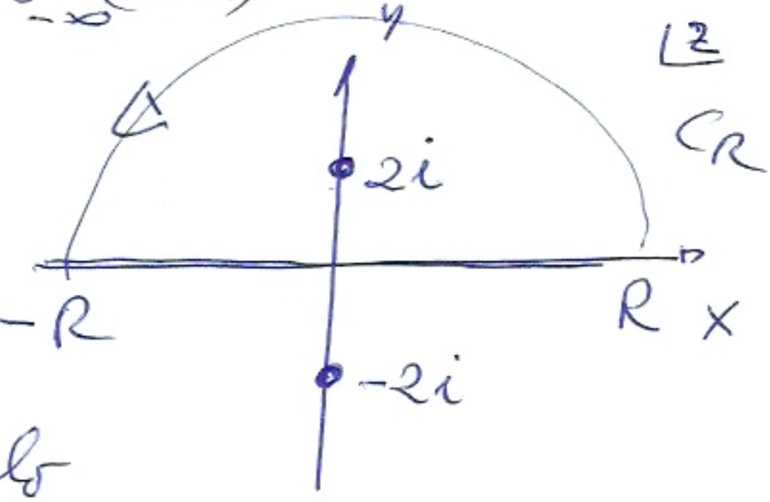
$$\oint_C f(z)\,dz = 2\pi i\left(\operatorname{Res} f(0) + \operatorname{Res} f(-4)\right) = 2\pi i\left(\frac{1}{4^3} - \frac{1}{4^3}\right) = 0$$

$$\boxed{\oint_C f(z)\,dz = 0}$$

Ex 3

$$\int_0^\infty \frac{x^2}{(x^2+4)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2}\,dx$$

$$f(z) = \frac{z^2}{(z^2+4)^2}.$$



$z = 2i$ polo duplo

$$\operatorname{Res} f(2i) = \lim_{z \to 2i} \frac{d}{dz}\left(\frac{z^2}{(z+2i)^2}\right)$$

$$\left(\frac{z^2}{(z+2i)^2}\right)' = \frac{2z}{(z+2i)^2} - \frac{z^2 2}{(z+2i)^3} = \frac{2z(z+2i) - 2z^2}{(z+2i)^3} =$$

$$= \frac{\cancel{2z^2} + 4iz - \cancel{2z^2}}{(z+2i)^3} = \frac{4iz}{(z+2i)^3}$$

$$\operatorname{Res} f(2i) = \frac{4i\,2i}{(4i)^3} = \frac{2i}{(4i)^2} = -\frac{2}{16}i = -\frac{1}{8}i$$

$$\oint_C f(z)\,dz = 2\pi i\left(-\frac{1}{8}i\right) = \frac{\pi}{4}$$

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = \frac{\pi}{4}$$

$$\boxed{\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = \frac{\pi}{4} - \lim_{R\to\infty}\int_{C_R} f(z)\,dz}$$

demonstração que o último termo é zero.

$$\left|\int_{C_R} f(z)\,dz\right| \le \int_{C_R} |f(z)|\,|dz| = \int_{C_R} \frac{|z|^2}{|z^2+4|^2}\,|dz| \le$$

desigualdade Triangular

$$\le \int_{C_R} \frac{|z|^2}{\left||z|^2-4\right|^2}\,|dz| =$$

$$= \frac{R^2}{(R^2-4)^2}\int_{C_R} |dz| = \frac{\pi R^3}{(R^2-4)^2}$$

$$\lim_{R\to\infty} \frac{\pi R^3}{(R^2-4)^2} = 0$$

$$\Rightarrow \lim_{R\to\infty}\left|\int_{C_R} f(z)\,dz\right| \le 0 \Longrightarrow$$

$$\left|\int_{C_R} f(z)\,dz\right| = 0$$ (porque o módulo só pode ser positivo)

então

$$\boxed{\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0}$$

obtemos que

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2}\, dx = \frac{\pi}{4}$$

$$\Rightarrow \boxed{\int_0^{\infty} \frac{x^2}{(x^2+4)^2}\, dx = \frac{\pi}{8}}$$